# A Time-Based Sampling Framework for Finite-Rate-of-Innovation Signals

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SPECTRUM LAB

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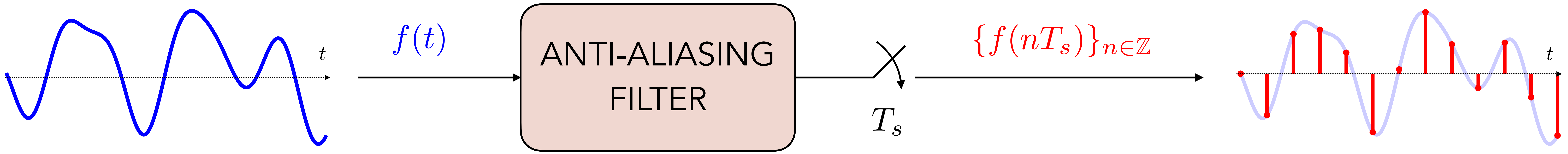
# Sampling Signals

## Uniform Sampling of Bandlimited Signals




- Finite-energy functions in bandlimited spaces: $f \in (L^2 \cap B_\Omega)(\mathbb{R})$ :

$$\boxed{f(t) = \sum_{n\in\mathbb{Z}} f(nT_s)\,\text{sinc}\left(\frac{t-nT_s}{T_s}\right).}$$

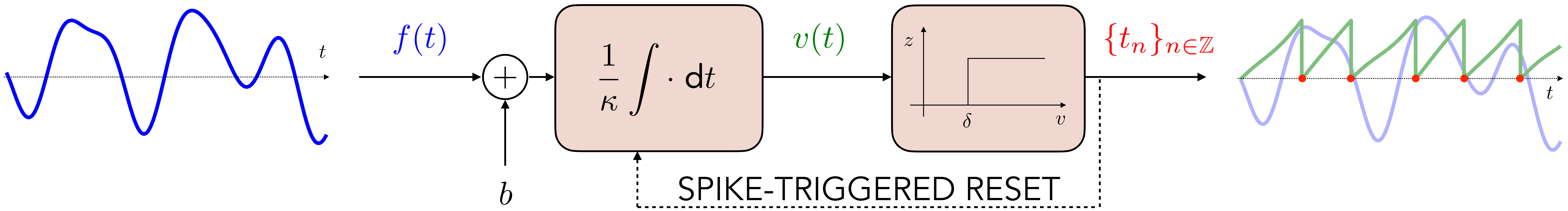
*Shannon's Sampling Theorem*

- The sequence of samples $\{f(nT_s)\}_{n\in\mathbb{Z}}$ completely specify the signal.

A.A. Lazar, "Time encoding with an integrate-and-fire neutron with a refractory period," *Neurocomput.*, 2004.

# Sampling Signals

## The Integrate-and-Fire Time-Encoding Machine (IF-TEM)




- The output of the time-encoding machine is a strictly increasing set of time-instants that follow:

$$\text{Stability: } \frac{\kappa\delta}{b + \max_{t\in\mathbb{R}} |f(t)|} < t_{n+1} - t_n < \frac{\kappa\delta}{b - \max_{t\in\mathbb{R}} |f(t)|},$$

$$\text{t-Transform: } \int_{t_n}^{t_{n+1}} f(t)\mathrm{d}t = -b(t_{n+1} - t_n) + \kappa\delta.$$

- The signal must be reconstructed from the sequence of samples $\{t_n\}_{n\in\mathbb{Z}}$.

A.A. Lazar, "Time encoding with an integrate-and-fire neutron with a refractory period," *Neurocomput.*, 2004.

# Time-Based Sampling

## Advantages

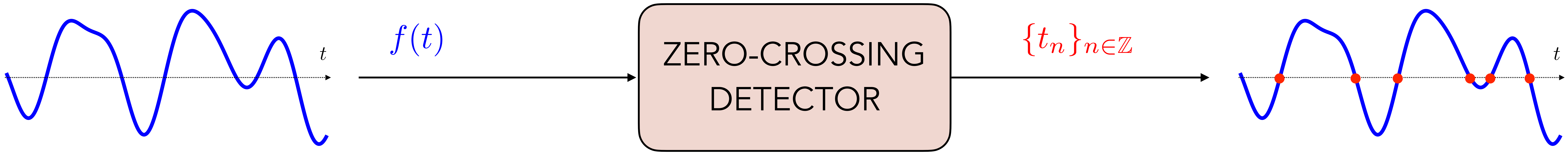
- Time encoding mimics representation of sensory signals.
- Time-encoding machine is an asynchronous device $\rightsquigarrow$ low power consumption.
- Nonuniform sampling $\rightsquigarrow$ sparse measurements.
- Event driven sampling $\rightsquigarrow$ no redundancy.

## Disadvantages

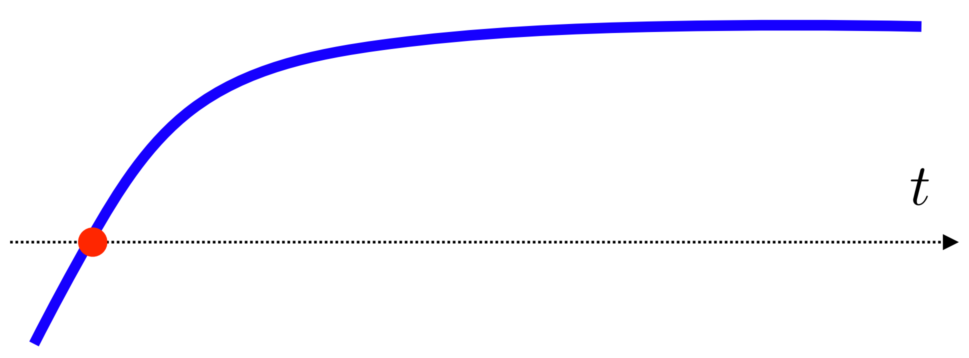
- Sophisticated sampling devices.
- Digital processing of continuous-domain signal is not possible.
- Iterative reconstruction techniques.

# Examples of Time-Encoding Machines

## Zero-Crossing Instants

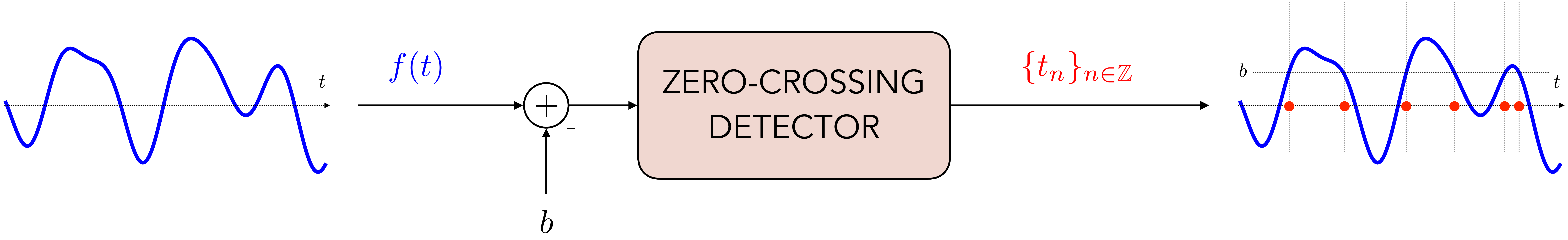



- Simple implementation.
- Signal might have very few zero-crossings.

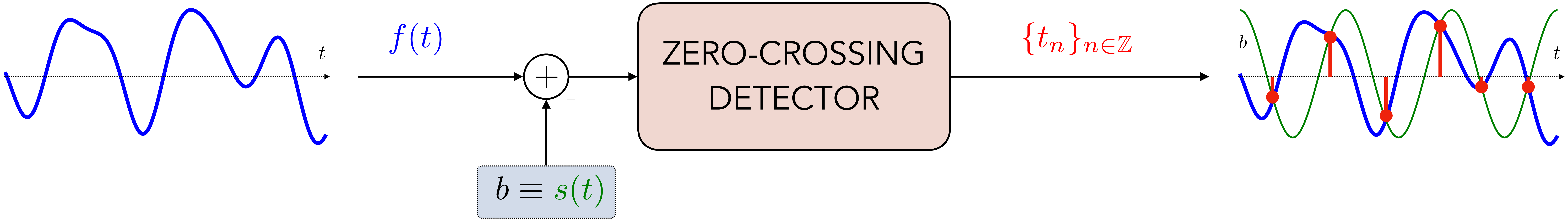


# Examples of Time-Encoding Machines

## Level-Crossing Instants




## Crossing-Time-Encoding Machine [Gontier, '14]




D. Gontier and M. Vetterli, "Sampling based on timing: Time encoding machines on shift-invariant subspaces," *Appl. Comput. Harmon. Anal.*, 2014.

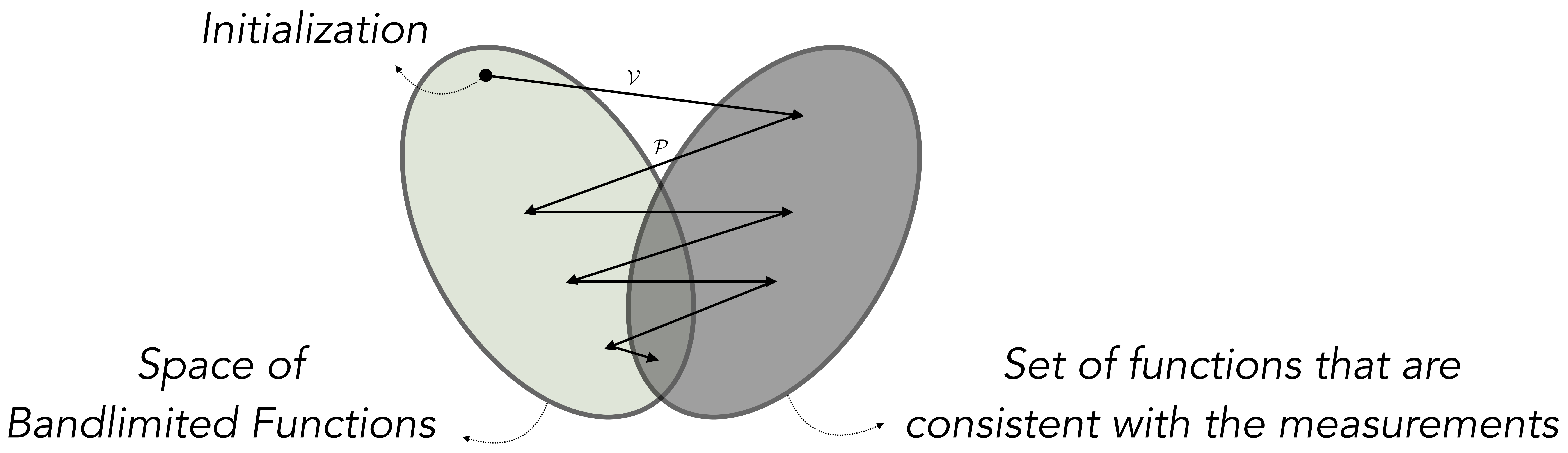
# Recovery of Bandlimited Signals

**Theorem 1: Alternating Projections [Wiley, '78]**

Let $f \in (L^2 \cap B_\Omega)(\mathbb{R})$ and let $\mathcal{P}$ the projection operator from $L^2(\mathbb{R})$ to $(L^2 \cap B_\Omega)(\mathbb{R})$. Then, $f$ can be recovered from its nonuniform samples $\{t_n\}_{n\in\mathbb{Z}}$ with $\max\limits_{n\in\mathbb{Z}} |t_{n+1} - t_n| < \dfrac{\pi}{\Omega}$ by the iterative algorithm:

$$f_{\ell+1} = \mathcal{P}\mathcal{V}f + (\mathcal{I} - \mathcal{P}\mathcal{V})f_\ell,$$

where $\mathcal{V}f = \sum\limits_{n\in\mathbb{Z}} f(t_n)\mathbb{1}_{[t_n, t_{n+1}[}(t)$.




R. G. Wiley, "Recovery of bandlimited signals from unequally spaced samples," *IEEE Trans. Comm.*, 1978.

# Recovery of Bandlimited Signals

**Theorem 2: Operator Formulation [Lazar, '04]**

Let $f \in (L^2 \cap B_\Omega)(\mathbb{R})$ and suppose $\max_{n\in\mathbb{Z}} |t_{n+1} - t_n| < \frac{\pi}{\Omega}$. Then, the input $f$ to the IF-TEM can be perfectly recovered as $f(t) = \lim_{\ell\to\infty} f_\ell(t)$, where

$$f_{\ell+1} = f_\ell + \mathcal{A}(f - f_\ell).$$

The operator $\mathcal{A}$ maps $f$ onto $(L^2 \cap B_\Omega)(\mathbb{R})$ as $\mathcal{A}f = \sum_{n\in\mathbb{Z}} \left[ \int_{t_n}^{t_{n+1}} f(t)\mathrm{d}t \right] \frac{\sin(\Omega t)}{\pi t}$.

*Generalizations to Shift-Invariant Spaces [Gontier, '14]*

*Extension to Multichannel Bandlimited Sampling [Adam, '19]*


A.A. Lazar, "Time encoding with an integrate-and-fire neuron with a refractory period," *Neurocomput.*, 2004.

D. Gontier and M. Vetterli, "Sampling based on timing: Time encoding machines on shift-invariant subspaces," *Appl. Comput. Harmon. Anal.*, 2014.

K. Adam et al., Sampling and reconstruction of bandlimited signals with multi-channel time encoding," *arXiv*, 2019.

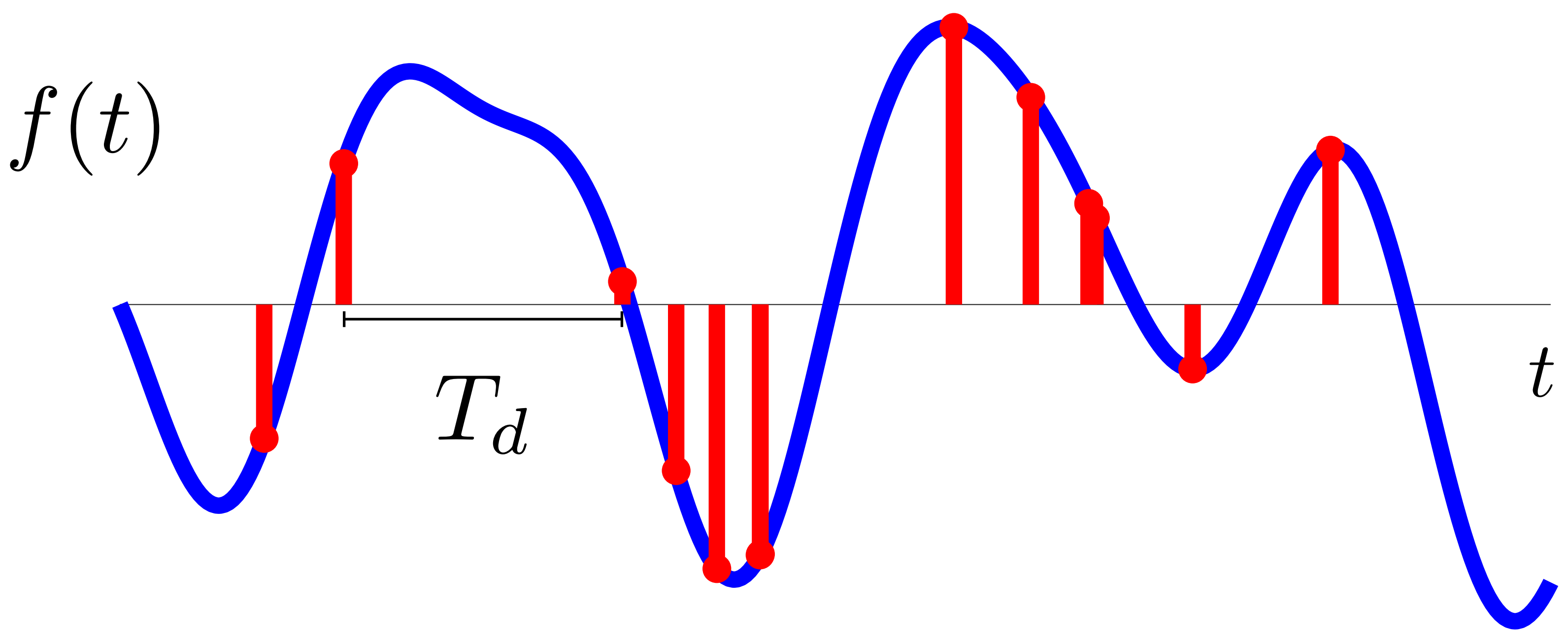
# Time-Encoding and Non-Uniform Sampling

## How to reconstruct from nonuniform samples?

### Bandlimited Signals: Alternating Projections

$$T_d = \max_{n \in \mathbb{Z}} |t_{n+1} - t_n| < \frac{\pi}{\Omega}$$

$$\Rightarrow \quad f_{\ell+1} = \mathcal{PV}f + (\mathcal{I} - \mathcal{PV})f_\ell \longrightarrow f$$




## How to guarantee dense sampling?

### Time-Encoding Machines

$$f \longmapsto \{t_n\}_{n \in \mathbb{Z}}$$

$$-\infty < \cdots < t_{-1} < t_0 < t_1 < \cdots < \infty$$

**Bounds on sampling density:**

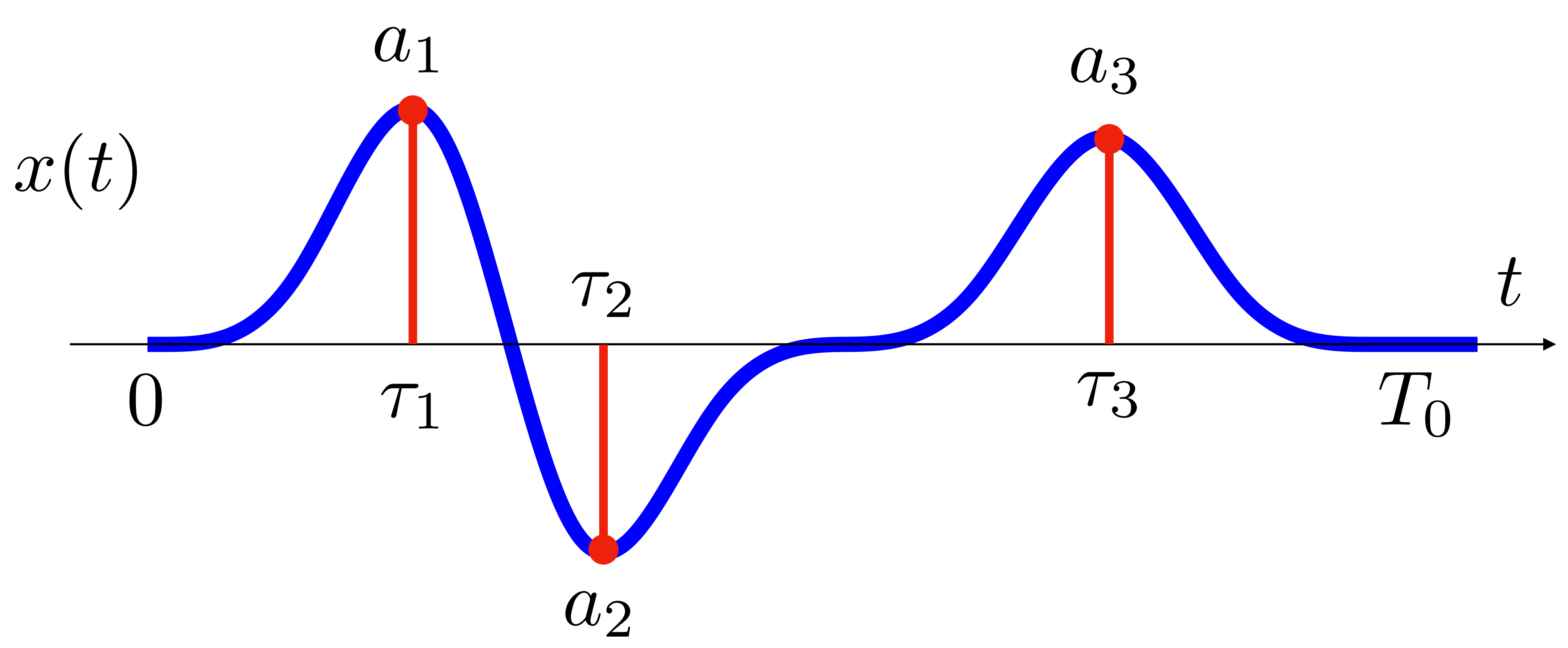
- Crossing TEM with $s(t) = A\cos(2\pi t / T_s)$ : $T_d < T_s$.
- Integrate-and-Fire TEM: $T_d < \dfrac{\kappa\delta}{b - \max_{t \in \mathbb{R}} |f(t)|}$.


R.J. Duffin and A.C. Schaeffer, "A class of nonharmonic Fourier series," *IEEE Trans. Amer. Math. Soc.*, 1952.

A. Aldroubi and K. Gröchenig, "Nonuniform sampling and reconstruction in shift-invariant spaces," *SIAM Review*, 2001.

# This Paper

- We consider time encoding of finite-rate-of-innovation (FRI) signals.
- In particular, we consider periodic sum of weighted and time-shifted pulses, and their time encoding using the C-TEM and IF-TEM.



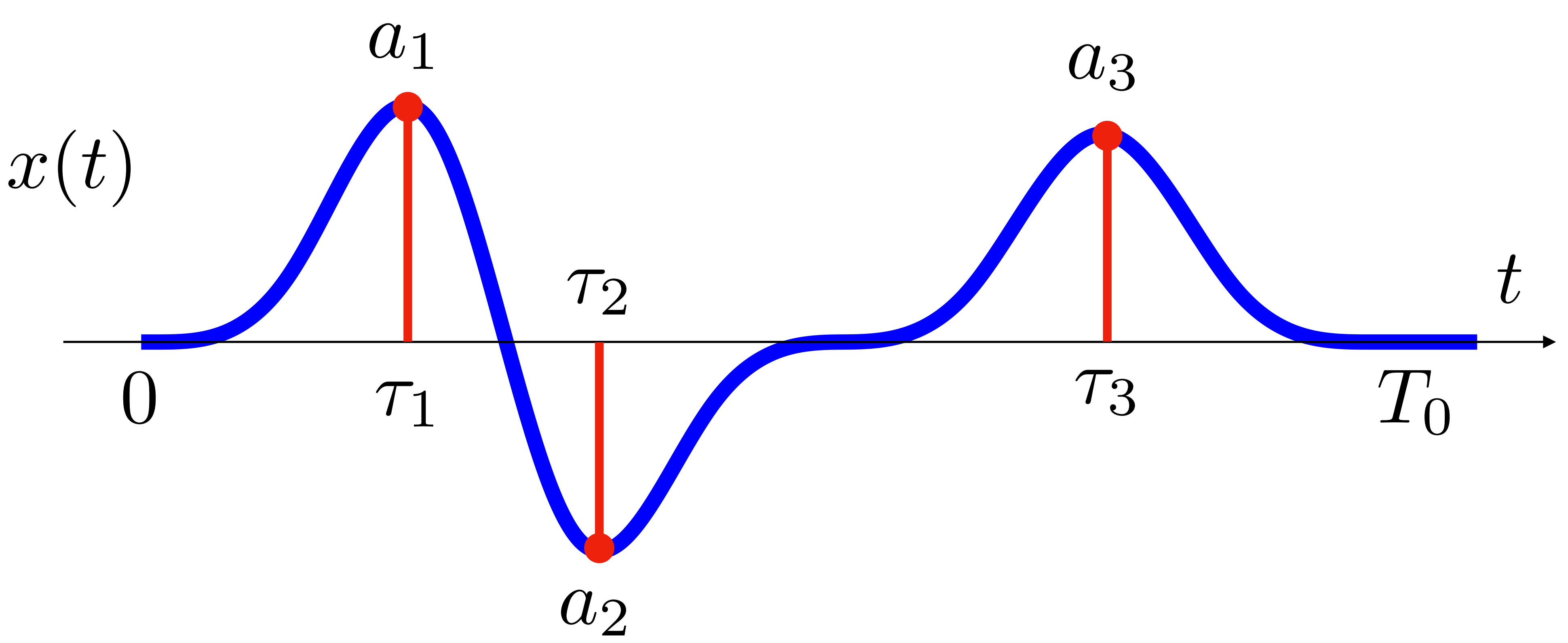

- Is reconstruction possible?
- If yes, under what conditions?

# Signal Model

## Sum of Weighted and Time Shifted Pulses

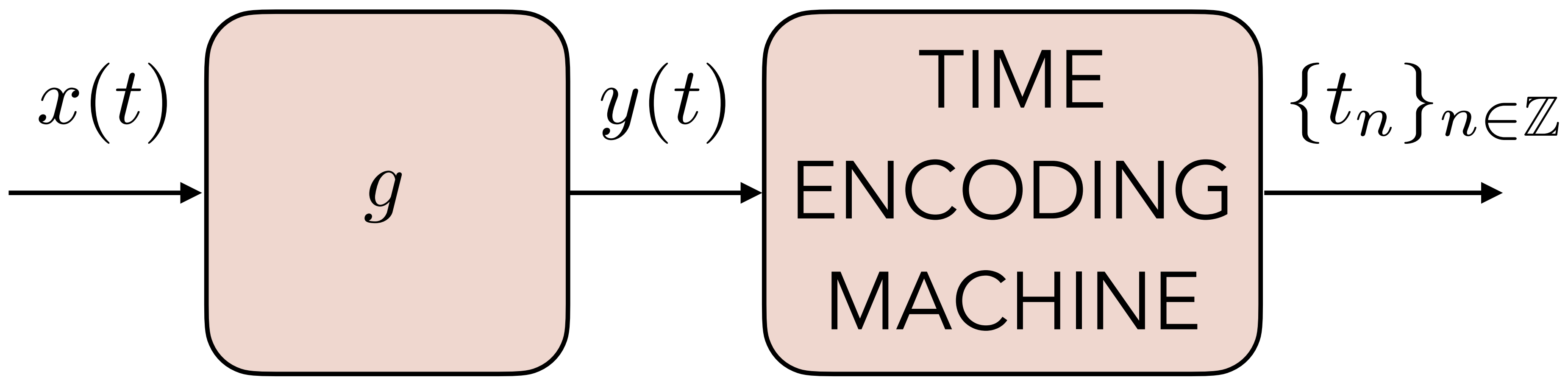
- Consider a $T_0$-periodic FRI signal, $x \in L^2([0, T_0[)$:

$$x(t) = \sum_{p \in \mathbb{Z}} \sum_{\ell=1}^{L} a_\ell h(t - \tau_\ell - pT_0)$$




- The pulse $h$ is known a priori.
- The parameters $\{a_\ell, \tau_\ell\}_{\ell=1}^{L}$ completely specify $x \rightsquigarrow x$ has finite rate of innovation.
- The rate of innovation of $x$ is $\dfrac{2L}{T_0} \rightsquigarrow x$ must be recoverable using $2L + 1$ measurements.

# Prior Art: Time Encoding of Impulse Streams




- Consider a stream of Dirac impulses: $x(t) = \sum_{\ell=1}^{L} a_\ell \delta(t - \tau_\ell)$
- Perfect reconstruction is guaranteed when:
  1. The sampling kernel $g$ is a first-order exponential spline.
  2. The sampling kernel must satisfy:

$$|\mathrm{supp}(g)| < \min_{\ell=1,\cdots,L} |\tau_{\ell+1} - \tau_\ell|.$$

- Reconstruction is sequential, i.e., each $a_\ell \delta(t - \tau_\ell)$ is reconstructed using signal moments:

$$s_m = \sum_{n=1}^{2} c_{m,n}^{I} y(t_n) = a_\ell e^{\alpha_m \tau_\ell}, \ m = 0, 1.$$

- This method can be extended to multi-channel bursts of Dirac impulses and to signals of the type $x(t) = \sum_{\ell=1}^{L} a_\ell h(t - \tau_\ell)$ with $h$ following some conditions.


R. Alexandru and P. L. Dragotti, "Reconstructing classes of non-bandlimited signals from time-encoded information," *arXiv*, 2019.

# Frequency-Domain Representation

- Since $x$ is $T_0$-periodic, it has a Fourier series representation:

$$x(t) = \sum_{k\in\mathbb{Z}} \hat{x}[k] e^{\mathrm{j}k\omega_0 t}, \;\; \omega_0 = \frac{2\pi}{T_0}$$

$$\text{where } \hat{x}[k] = \frac{1}{T_0}\hat{h}(k\omega_0) \underbrace{\boxed{\sum_{\ell=1}^{L} a_\ell e^{-\mathrm{j}k\omega_0\tau_\ell}}}_{\hat{r}[k]}.$$

*Sum of Weighted Complex Exponentials (SWCE)*

- $2L+1$ contiguous samples of $\hat{x}[k]$ are sufficient for parameter estimation [Vetterli, '02].
- The annihilating filter $\{\gamma\}_{\ell=0}^{L}$ has the Z-transform:

$$\Gamma(z) = \prod_{\ell=1}^{L} (1 - e^{-\mathrm{j}\omega_0\tau_\ell})$$

- The filter annihilates $\hat{r}$:

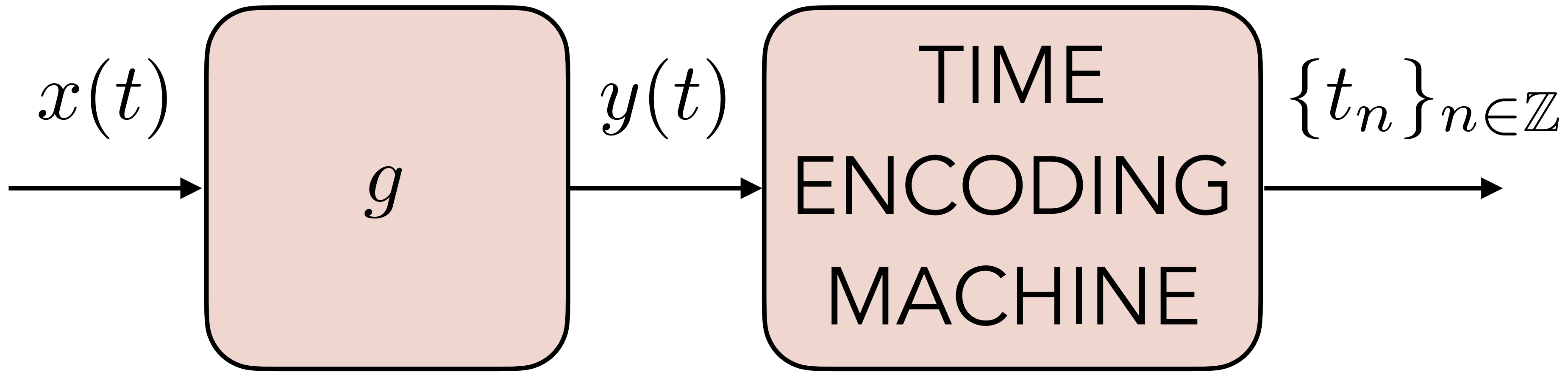
$$\mathbf{R}\gamma = \mathbf{0}$$

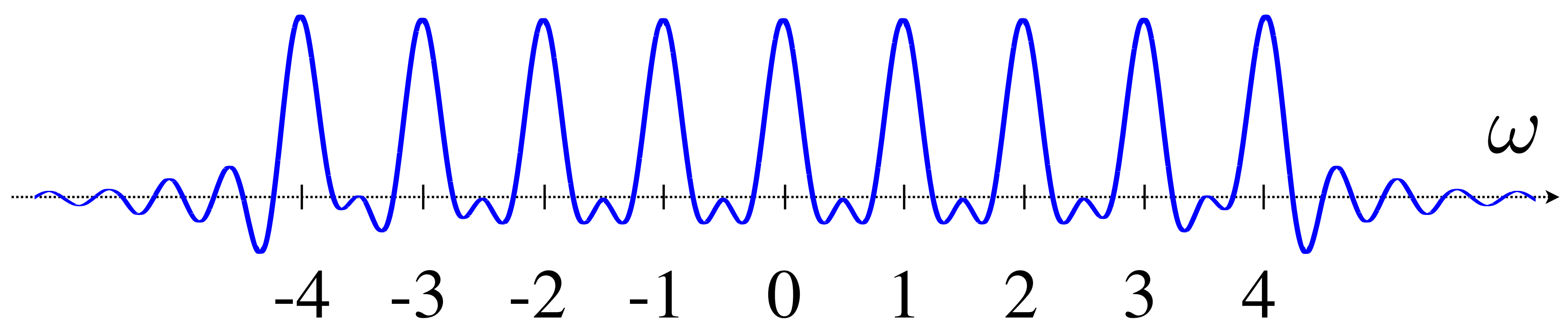
M. Vetterli et al., "Sampling signals with finite rate of innovation," *IEEE Trans. Signal Process.*, 2002.

# Kernel-Based Sampling of FRI Signals using TEMs

- The filtered signal $y$ is $T_0$-periodic:

$$
\begin{aligned}
y(t) = (x * g)(t) &= \int_{-\infty}^{\infty} x(\nu) g(t-\nu) \mathrm{d}\nu, \\
&= \sum_{k\in\mathbb{Z}} \hat{x}[k] \int_{-\infty}^{\infty} g(t-\nu)\, e^{\mathrm{j}k\omega_0\nu} \mathrm{d}\nu, \\
&= \sum_{k\in\mathbb{Z}} \hat{x}[k]\, \hat{g}(k\omega_0)\, e^{\mathrm{j}k\omega_0 t}.
\end{aligned}
$$




- Let the sampling kernel $g$ satisfy alias-cancellation conditions [Tur, '11]

$$
\hat{g}(k\omega_0) = \begin{cases} 1, & k \in \mathcal{K}, \\ 0, & k \notin \mathcal{K}, \end{cases}
$$

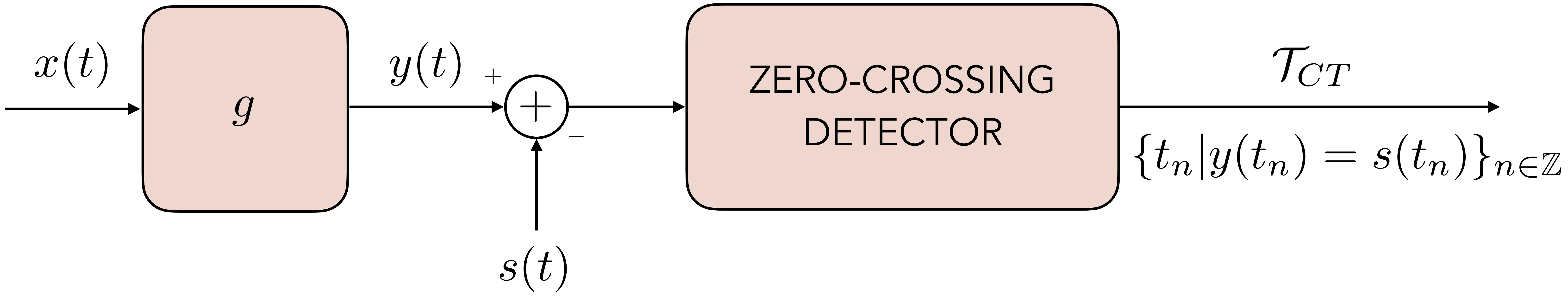
where $\mathcal{K} = \{-K, \cdots, -1, 0, 1, \cdots, K\}$, for some $K \in \mathbb{N}$.

- The filtered signal is a trigonometric polynomial $\boxed{y(t) = \sum_{k\in\mathcal{K}} \hat{x}[k] e^{\mathrm{j}k\omega_0 t}}$.

*A Finite Sum*

R. Tur et al., "Innovation rate sampling of pulse streams with applications to ultrasound imaging," *IEEE Trans. Signal Process.*, 2011.

# Sampling using Crossing-Time-Encoding Machine




*Linear System of Equations*

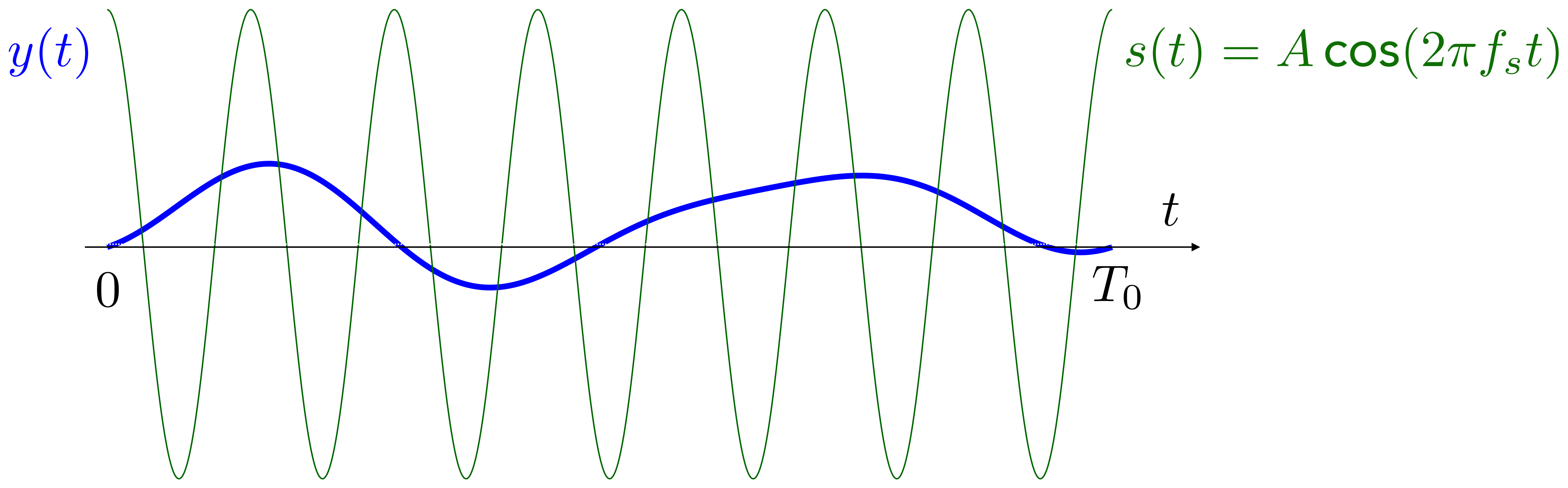
$$y(t_n) = \sum_{k\in\mathcal{K}} \hat{x}[k]\, e^{\mathrm{j}k\omega_0 t_n}$$

$$\underbrace{\begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_N) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} e^{-\mathrm{j}K\omega_0 t_1} & \dots & e^{-\mathrm{j}\omega_0 t_1} & 1 & e^{\mathrm{j}\omega_0 t_1} & \dots & e^{\mathrm{j}K\omega_0 t_1} \\ e^{-\mathrm{j}K\omega_0 t_2} & \dots & e^{-\mathrm{j}\omega_0 t_2} & 1 & e^{\mathrm{j}\omega_0 t_2} & \dots & e^{\mathrm{j}K\omega_0 t_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ e^{-\mathrm{j}K\omega_0 t_N} & \dots & e^{-\mathrm{j}\omega_0 t_N} & 1 & e^{\mathrm{j}\omega_0 t_N} & \dots & e^{\mathrm{j}K\omega_0 t_N} \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} \hat{x}[-K] \\ \vdots \\ \hat{x}[-1] \\ \hat{x}[0] \\ \hat{x}[1] \\ \vdots \\ \hat{x}[K] \end{bmatrix}}_{\mathbf{x}}$$

# Sufficient Conditions for Perfect Recovery

## Recovery from C-TEM Measurements

- For recovery of parameters $\{a_\ell, \tau_\ell\}_{\ell=1}^{L}$ using the annihilating filter, $|\mathcal{K}| \geq 2L+1$, hence $N \geq 2L+1$.
- The sinusoidal reference crosses the signal at least once every period whenever $|A| \geq \sup_{t \in [0, T_0[} |y(t)|$.
- Hence, to record $N \geq 2L+1$ samples in $T_0$-length interval, $f_s \geq \dfrac{2L+1}{T_0}$.

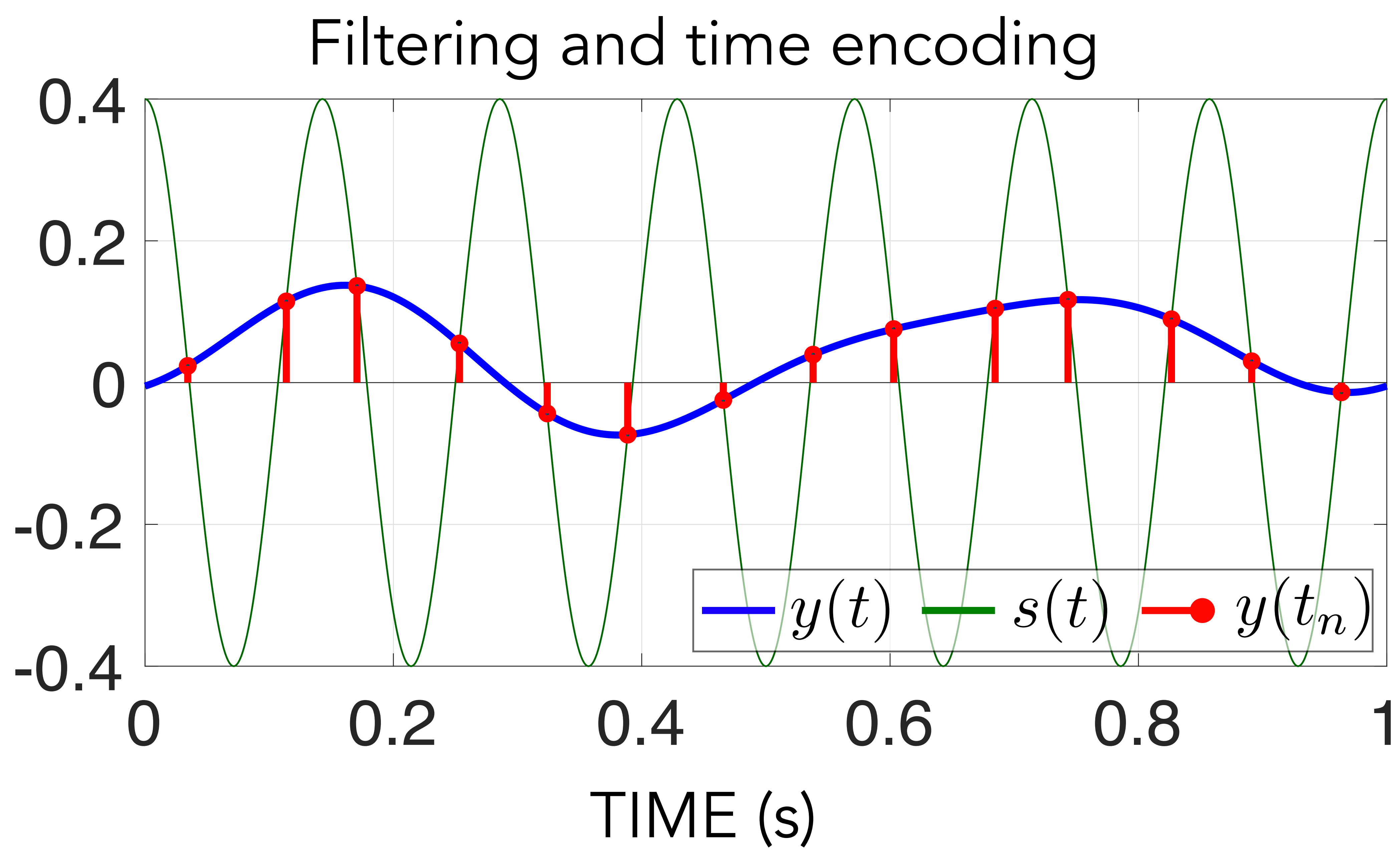


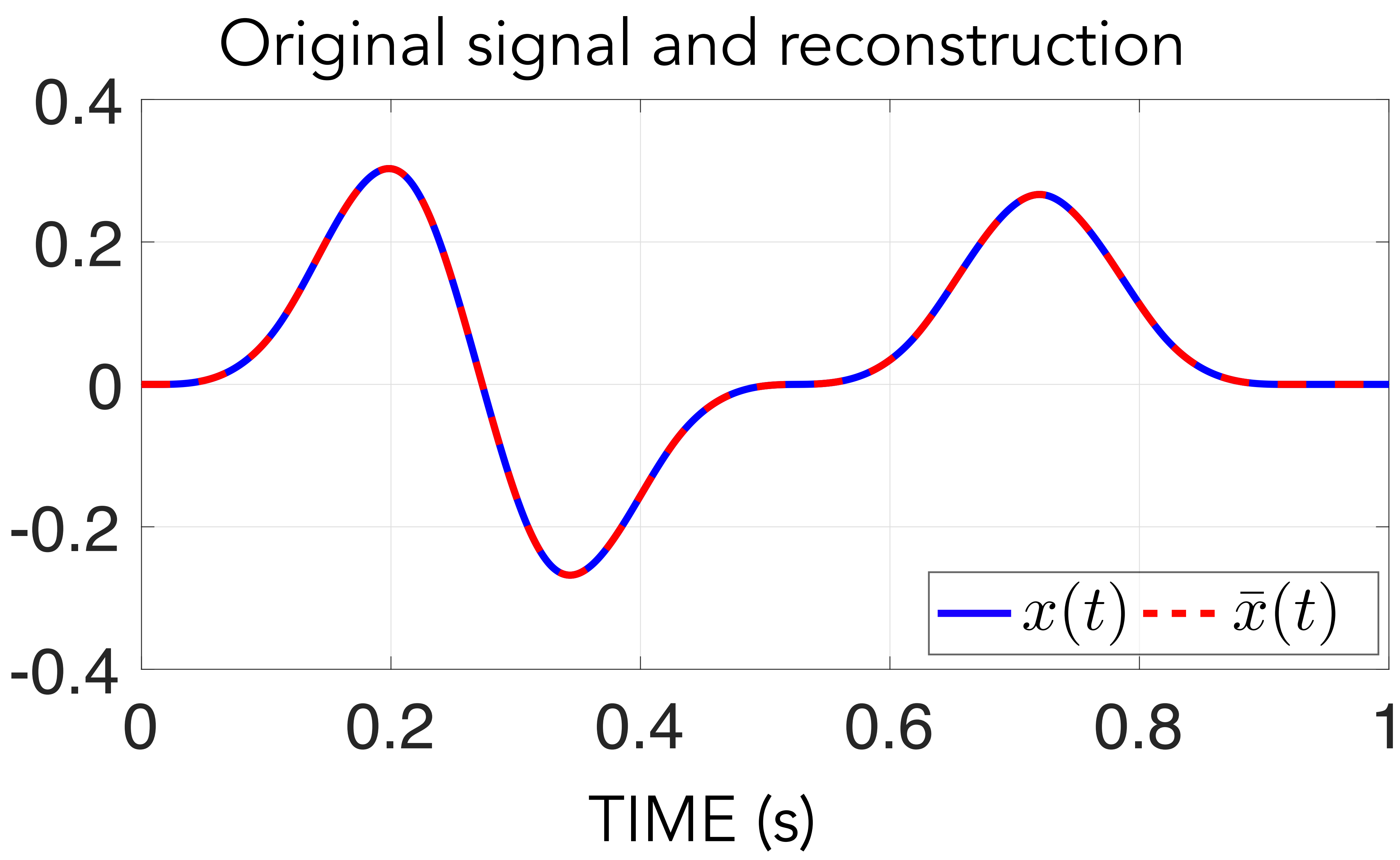
# Sufficient Conditions for Perfect Recovery

## Recovery from C-TEM Measurements

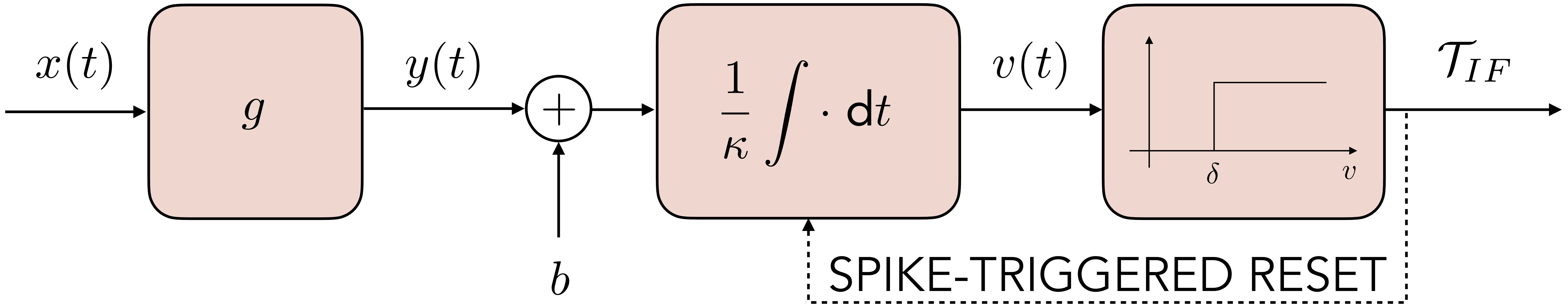
**Proposition 1: Recovery from Crossing-Time-Encoding Machine**

The set of time instants $\{t_n\}_{n=1}^{N} \subset \mathcal{T}_{CT}$ obtained using the C-TEM is a sufficient representation of the $T_0$-periodic signal $x$ with $N \geq 2L+1$, when the reference signal $s(t) = A\cos(2\pi f_s t)$ satisfies $|A| > \sup_{t\in[0,T_0[} |y(t)|$ and $f_s \geq \dfrac{2L+1}{T_0}$.



# Sampling using Integrate-and-Fire TEM

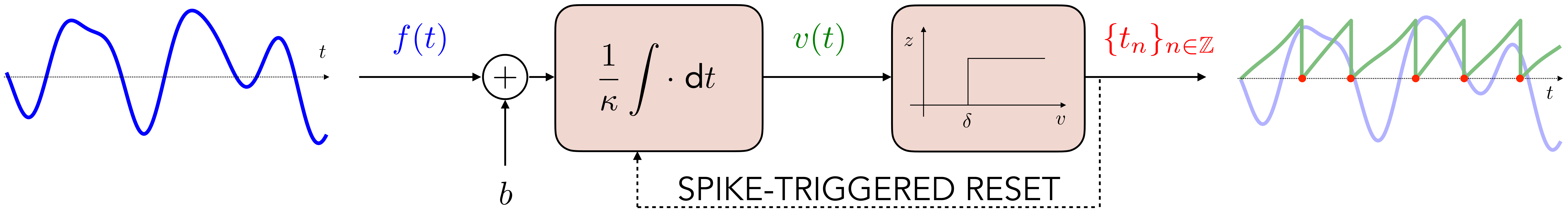



$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} y(t)\mathrm{d}t &= \int_{t_{t_n}}^{t_{n+1}} \sum_{k\in\mathcal{K}\setminus\{0\}} \hat{x}[k]e^{\mathrm{j}k\omega_0 t}\mathrm{d}t + \int_{t_{t_n}}^{t_{n+1}} \hat{x}[0]\mathrm{d}t \\
&= \sum_{k\in\mathcal{K}\setminus\{0\}} \frac{\hat{x}[k]}{\mathrm{j}k\omega_0}\left(e^{\mathrm{j}k\omega_0 t_{n+1}} - e^{\mathrm{j}k\omega_0 t_n}\right) + \hat{x}[0](t_{n+1}-t_n)
\end{aligned}
$$

$$
\begin{bmatrix} \int_{t_1}^{t_2} y(t)\mathrm{d}t \\ \int_{t_2}^{t_3} y(t)\mathrm{d}t \\ \vdots \\ \int_{t_{N-1}}^{t_N} y(t)\mathrm{d}t \end{bmatrix}
= \underbrace{\begin{bmatrix}
e^{-\mathrm{j}K\omega_0 t_2} - e^{-\mathrm{j}K\omega_0 t_1} & \cdots & t_2 - t_1 & \cdots & e^{\mathrm{j}K\omega_0 t_2} - e^{\mathrm{j}K\omega_0 t_1} \\
e^{-\mathrm{j}K\omega_0 t_3} - e^{-\mathrm{j}K\omega_0 t_2} & \cdots & t_3 - t_2 & \cdots & e^{\mathrm{j}K\omega_0 t_3} - e^{\mathrm{j}K\omega_0 t_2} \\
\vdots & & \vdots & & \vdots \\
e^{-\mathrm{j}K\omega_0 t_N} - e^{-\mathrm{j}K\omega_0 t_{N-1}} & \cdots & t_N - t_{N-1} & \cdots & e^{\mathrm{j}K\omega_0 t_N} - e^{\mathrm{j}K\omega_0 t_{N-1}}
\end{bmatrix}}_{\mathbf{Q}}
\begin{bmatrix} -\frac{\hat{x}[-K]}{\mathrm{j}K\omega_0} \\ \vdots \\ \hat{x}[0] \\ \vdots \\ \frac{\hat{x}[K]}{\mathrm{j}K\omega_0} \end{bmatrix}
$$

# Sampling Signals (revisited)

## The Integrate-and-Fire Time-Encoding Machine (IF-TEM)




- The output of the time-encoding machine is a strictly increasing set of time-instants that follow:

$$\text{Stability: } \frac{\kappa\delta}{b + \max_{t\in\mathbb{R}} |f(t)|} < t_{n+1} - t_n < \frac{\kappa\delta}{b - \max_{t\in\mathbb{R}} |f(t)|},$$

$$\text{t-Transform: } \int_{t_n}^{t_{n+1}} f(t)\mathrm{d}t = -b(t_{n+1} - t_n) + \kappa\delta.$$

- The signal must be reconstructed from the sequence of samples $\{t_n\}_{n\in\mathbb{Z}}$.


A.A. Lazar, "Time encoding with an integrate-and-fire neutron with a refractory period," *Neurocomput.*, 2004.

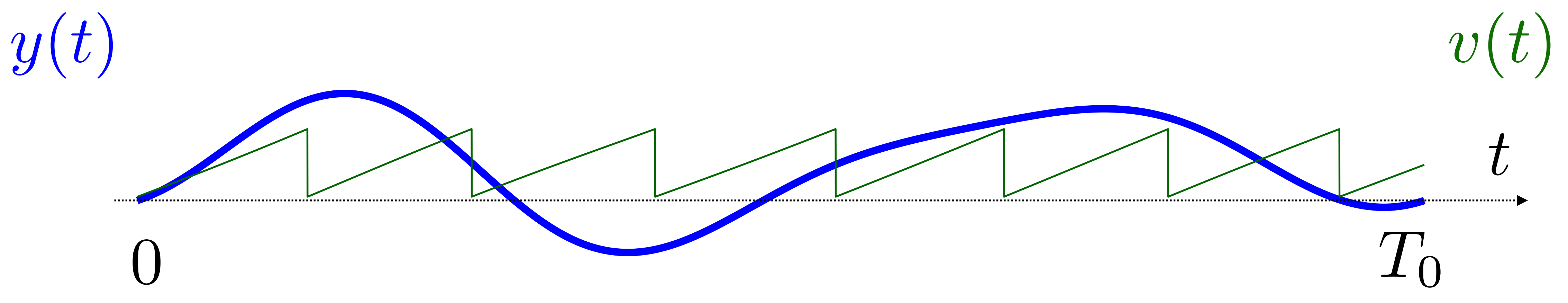
# Sufficient Conditions for Perfect Recovery

## Recovery from IFTEM Measurements

- For recovery of parameters $\{a_\ell, \tau_\ell\}_{\ell=1}^{L}$ using the annihilating filter, $|\mathcal{K}| \geq 2L+1$, i.e., $N-1 \geq 2L+1$.
- After the first trigger $t_1$, the further $N-1$ triggers must come up in the interval $[0, T_0[$ and using the upper bound on the difference:

$$t_1 + (N-1)\frac{\kappa\delta}{b - \sup\limits_{[0,T_0[} |y(t)|} < T_0.$$

- The maximum value of $t_1 \leq \dfrac{\kappa\delta}{b - \sup\limits_{[0,T_0[} |y(t)|}$.
- Hence the parameters must satisfy $\dfrac{\kappa\delta}{b - \sup\limits_{[0,T_0[}} < \dfrac{T_0}{N}$.

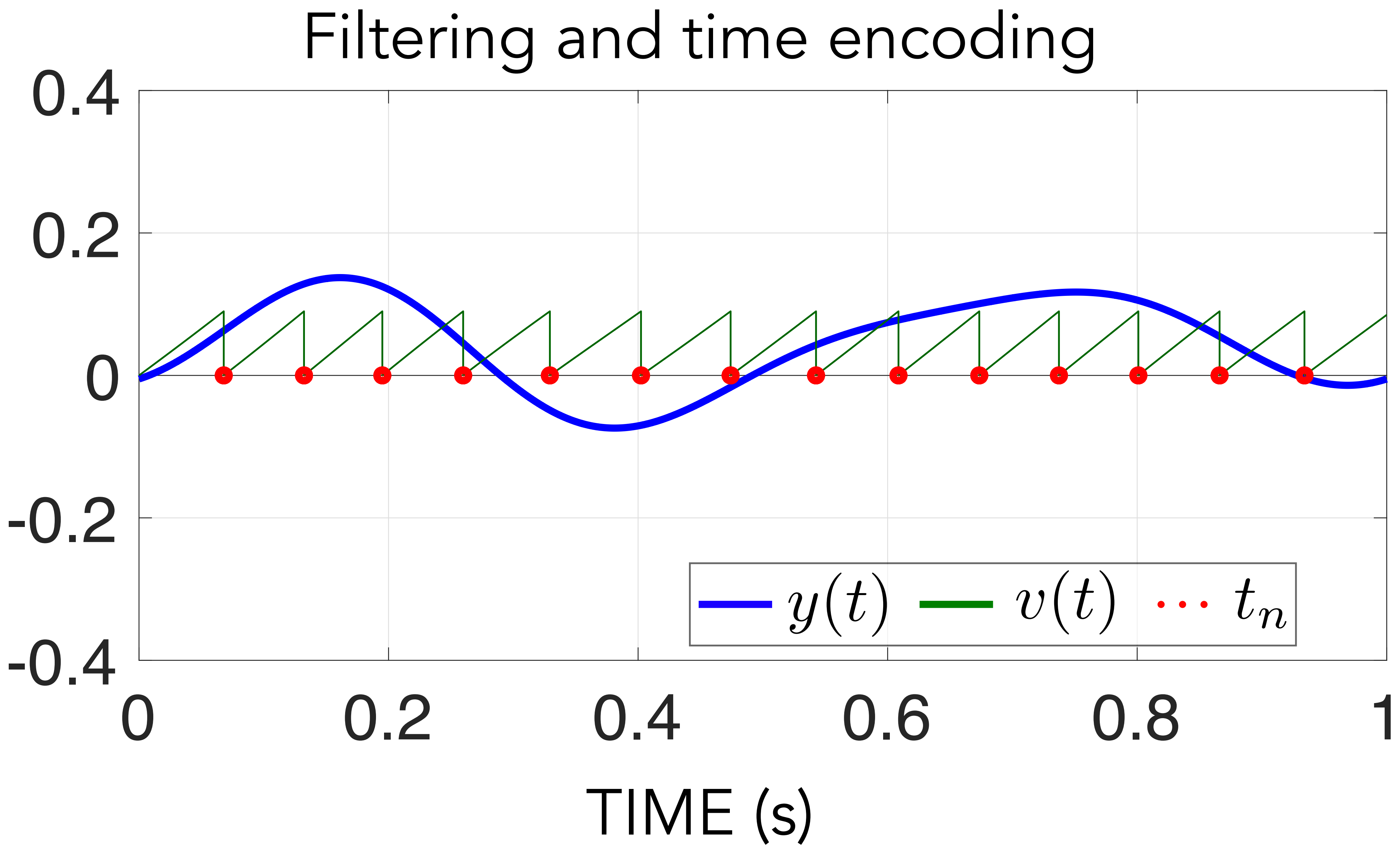


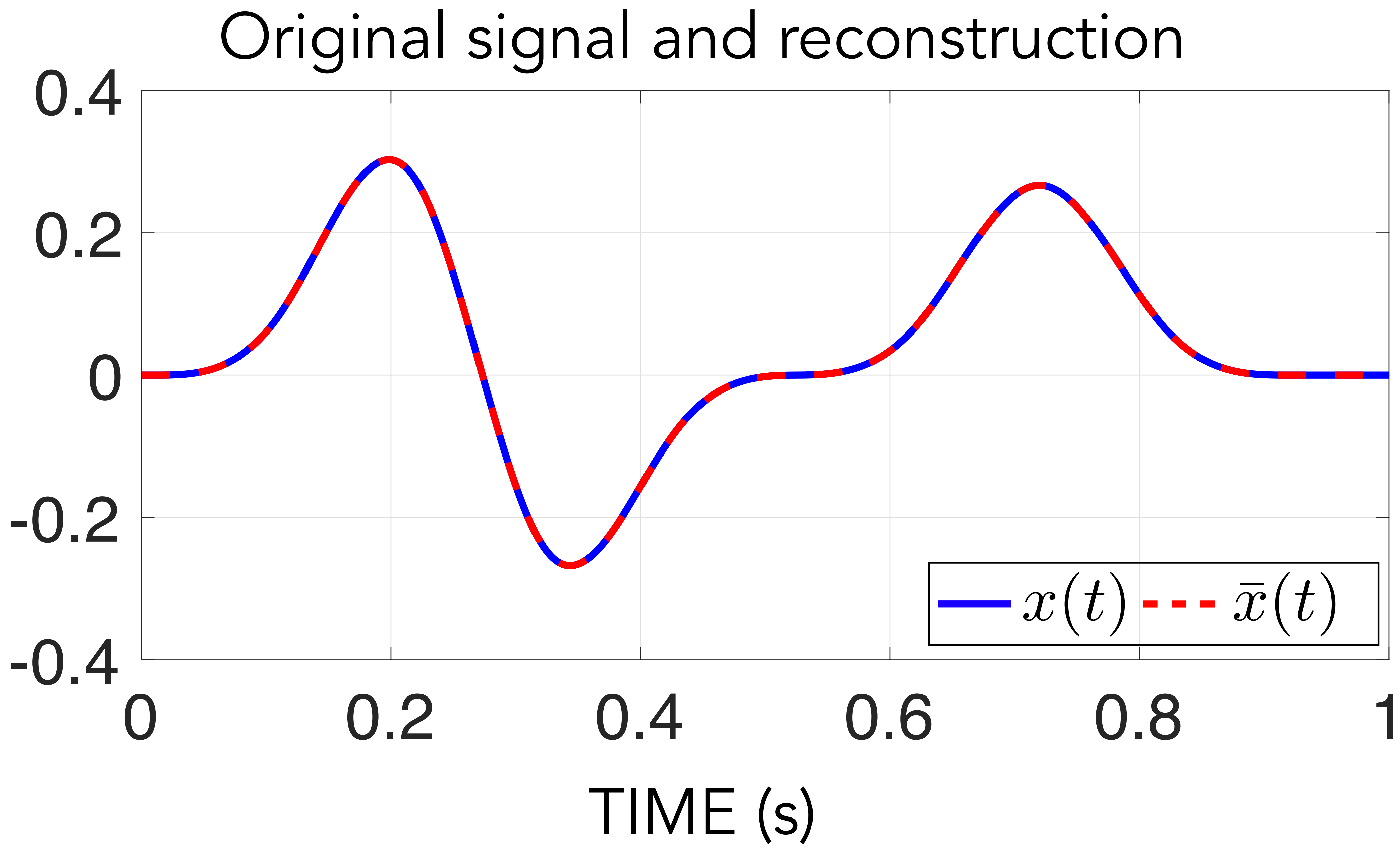
# Sufficient Conditions for Perfect Recovery

## Recovery from IFTEM Measurements

**Proposition 2: Recovery from Integrate-and-Fire Time-Encoding Machine**

The set of time instants $\{t_n\}_{n=1}^{N} \subset \mathcal{T}_{IF}$ obtained using the IF-TEM is a sufficient representation of the $T_0$-periodic signal $x$ when, the matrix $\mathbf{Q}$ has full column rank and the parameters of the TEM satisfy the condition $\frac{\kappa\gamma}{(b - \sup_{[0,T_0[} |y(t)|)} < \frac{T_0}{N}$ with $N \geq 2(L+1)$.



# Extension to Aperiodic Signals

## Periodized Sampling Kernel

- Convolution of a periodic signal $x$ with a kernel $g$ is equivalent to convolution of one period of the signal $x$ with a periodized version of the kernel $g$.

- Further, if the pulse $h$ has finite support, the periodization reduces to a finite replication.

$$x(t) = \sum_{p \in \mathbb{Z}} \sum_{\ell=1}^{L} a_\ell h(t - \tau_\ell - pT_0) \longrightarrow \boxed{g(t)} \longrightarrow y(t)$$

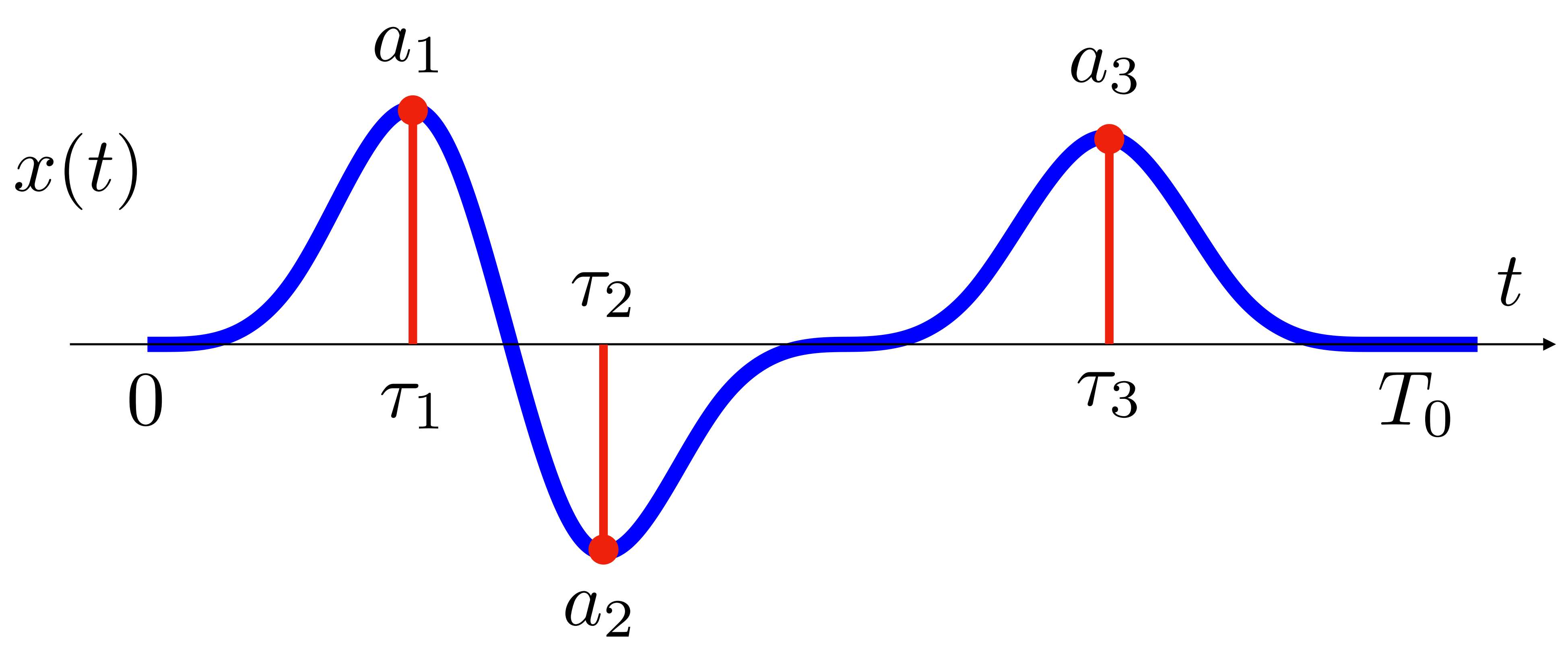
$$\tilde{x}(t) = \sum_{\ell=1}^{L} a_\ell h(t - \tau_\ell) \longrightarrow \boxed{\tilde{g}(t) = \sum_{p=-P}^{P} g(t - pT_0)} \longrightarrow y(t)$$

- The analysis of sampling and reconstruction of aperiodic-FRI signals is equivalent to those of periodic signals after these appropriate modifications to the sampling kernel.

R. Tur et al., "Innovation rate sampling of pulse streams with applications to ultrasound imaging," *IEEE Trans. Signal Process.*, 2011.

# Summary

- We considered time encoding of finite-rate-of-innovation (FRI) signals.
- In particular, we considered periodic sum of weighted and time-shifted pulses, and their time encoding using the C-TEM and IF-TEM.




- Is reconstruction possible?
  Yes. We showed parameter recovery is possible using frequency-domain analysis.
- If yes, under what conditions?
  We gave the sufficient conditions under which perfect reconstruction from C-TEM and IF-TEM measurements is possible.
- We also showed an extension of the theory to aperiodic sum of weighted and time-shifted signals.

# *Questions?*