FINITE RATE OF INNOVATION MODELLING OF FOURIER DESCRIPTORS

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1. Overview

- A finite-rate-of-innovation (FRI) model for closed contours using *Fourier descriptors*.
- Noise-robust estimation of Fourier descriptors (FDs) from partial measurements.
- Reconstruction of higher-order curves with $\mathcal{O}(N \log N)$ complexity.

2. Formulation

- For a closed contour $C : \{x(t), y(t)\}$, define $s(t) \triangleq x(t) + \mathrm{j}y(t)$, such that:

$$s(t) = \sum_{k \in \mathbb{Z}} c_k \, e^{\mathrm{j}kt}, \; 0 \le t < 2\pi, \; c_k \in \mathbb{C}.$$

- Uniform samples of the coordinate functions:

$$x(nT) = \sum_{k=-K}^{K} \alpha_k e^{\mathrm{j}knT}, \quad y(nT) = \sum_{k=-K}^{K} \beta_k e^{\mathrm{j}knT}$$

with $\alpha_k = \alpha_{-k}$ and $\beta_k = -\beta_{-k}$ such that $\alpha_k + \mathrm{j}\beta_k = c_k$.

- Problem: Given the noisy measurements $\{\tilde{x}(t_n), \tilde{y}(t_n)\}_{n=1}^{N}$, estimate $\{c_k\}_{k=-K}^{K}$.

3. Parameter Estimation

- Estimation of T: *Block Annihilation*
  - Construct the convolution matrices $\mathbf{X}$ and $\mathbf{Y}$ from $\{\tilde{x}(nT)\}$ and $\{\tilde{y}(nT)\}$.
  - Find the filter $\mathbf{h}$ such that

$$\mathbf{h}^* = \arg\min_{\mathbf{h}} \left\| \begin{bmatrix} X \\ Y \end{bmatrix} \mathbf{h} \right\|_2^2, \text{ s. t. } \|\mathbf{h}\|_2^2 = 1.$$

  - Roots of the polynomial with the coefficients $\mathbf{h}$ are the estimates of $\{kT\}_{-K}^{K}$.
- Estimation of Fourier Descriptors: *FRI-FD*
  - The weights α_k, β_k are estimated as the using least squares solutions to $\mathbf{E}\alpha = \tilde{\mathbf{x}}$ and $\mathbf{E}\beta = \tilde{\mathbf{y}}$, where $\mathbf{E}$ is a Vandermonde matrix of complex exponentials given as:

$$\begin{bmatrix} e^{-\mathrm{j}KT} & \cdots & e^{-\mathrm{j}T} & 1 & e^{\mathrm{j}T} & \cdots & e^{\mathrm{j}KT} \\ e^{-\mathrm{j}2KT} & \cdots & e^{-\mathrm{j}2T} & 1 & e^{\mathrm{j}2T} & \cdots & e^{\mathrm{j}2KT} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-\mathrm{j}NKT} & \cdots & e^{-\mathrm{j}NT} & 1 & e^{\mathrm{j}NT} & \cdots & e^{\mathrm{j}NKT} \end{bmatrix}.$$

4. Sampling Jitter and Denoising

- Non-uniform samples are modelled as sampling jitter: $t_n = nT + \nu_n$, where $\nu_n \overset{\text{i.i.d}}{\sim} \mathcal{U}\left[-\frac{T}{2}, \frac{T}{2}\right]$.
- The corresponding uniform samples have random amplitude modulated weights:

$$x_u(nT) = x(t_n) = \sum_{k=-K}^{K} \alpha_k e^{\mathrm{j}\nu_n} e^{\mathrm{j}knT}.$$

- Curve-specific information is in the interval $[-KT, KT]$.
- Convolution with an $M-$tap lowpass filter with cut-off frequency close to KT results in

$$(x_u * g)(nT) \approx \sum_{k=-K}^{K} G(kT)\alpha_k e^{\mathrm{j}knT}.$$

- Design the denoising filter $\{g(n)\}_{n=1}^{M}$ such that $G(kT) \approx 1, -K \le k \le K$.
  - A Tukey window with parameter 0.99 and $M = \left\lfloor \frac{2N}{3} \right\rfloor$ is used as the lowpass filter.

5. Noise Robustness

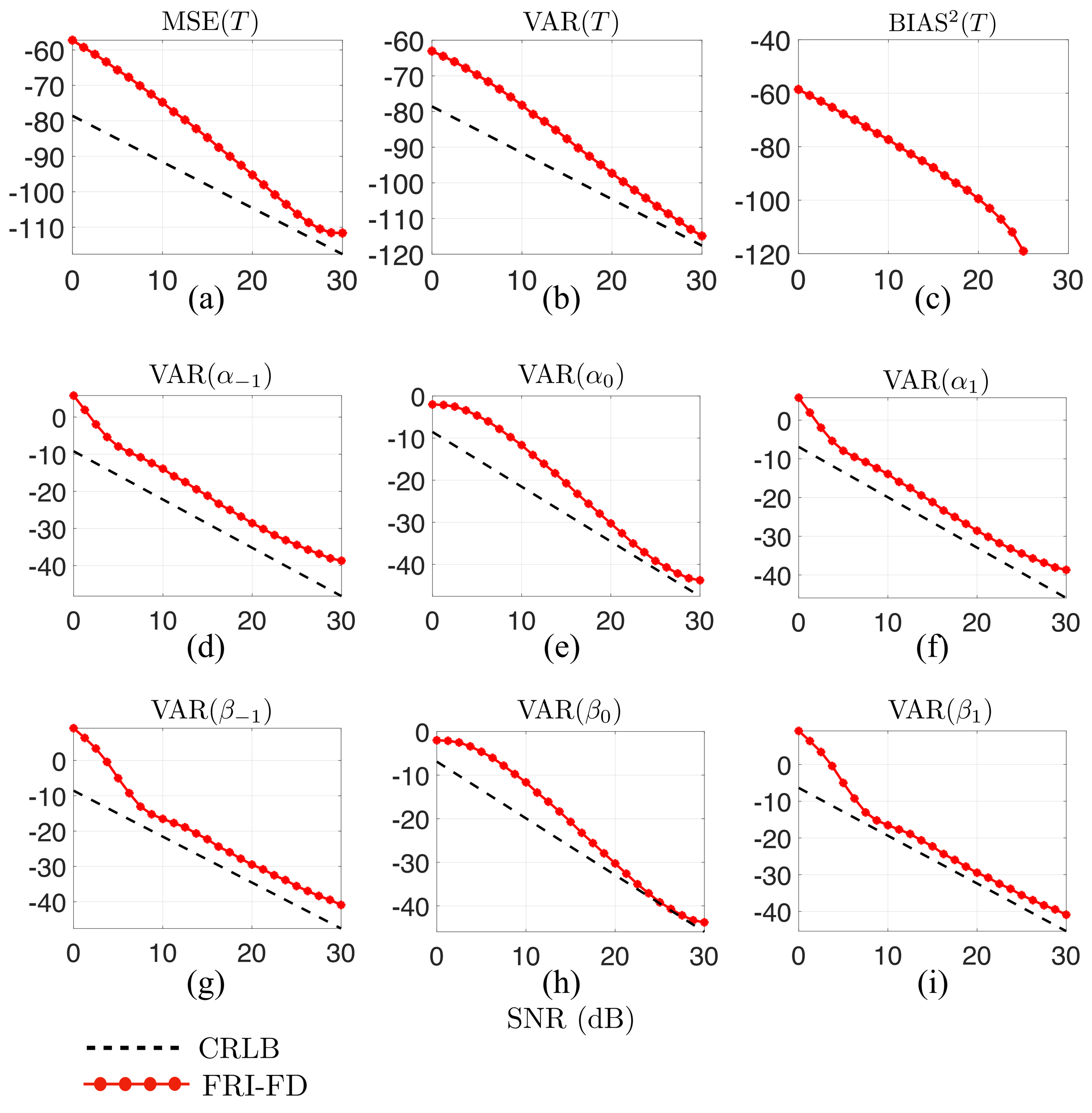
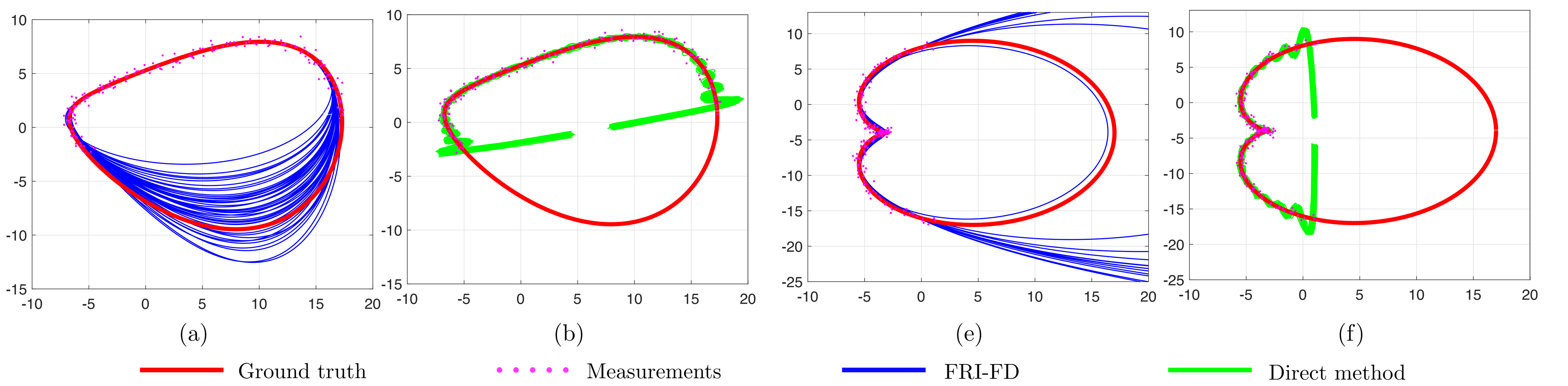


Figure 1: (a) Mean-square error (MSE); (b) Variance; and (c) Bias^2 in the estimation of T. (d-i) Variances in the estimation of FDs for a contour with $T = 0.01$, $\alpha_0 = 2$, $\alpha_{-1} = \alpha_1 = 8$, $\beta_0 = 3$, $\beta_{-1} = -\beta_1 = 7$. Results are obtained by averaging over 5000 independent Monte Carlo realizations.

6. Simulations



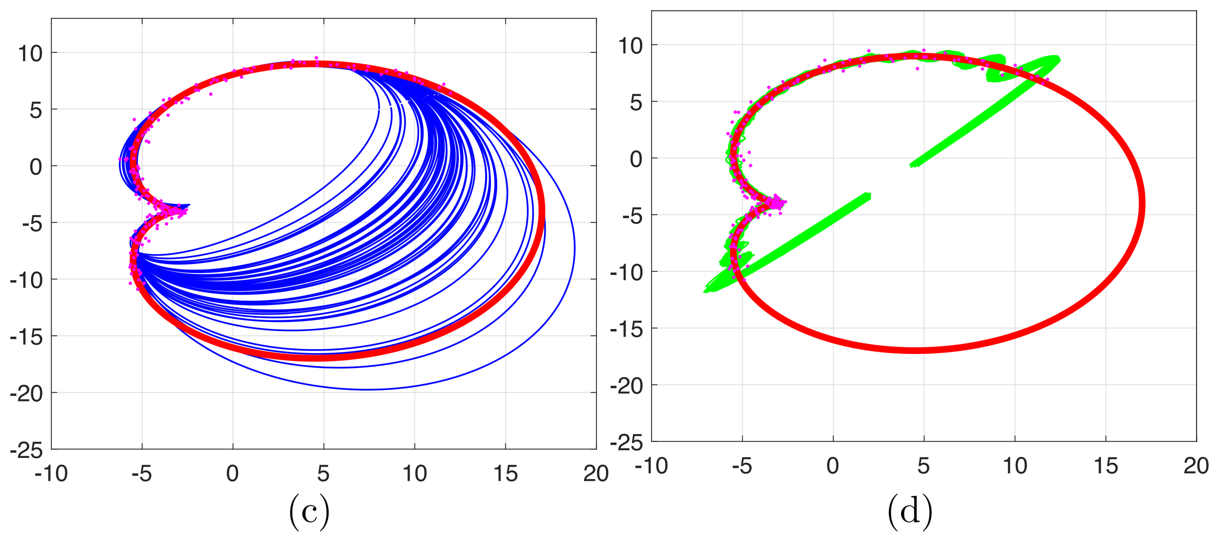
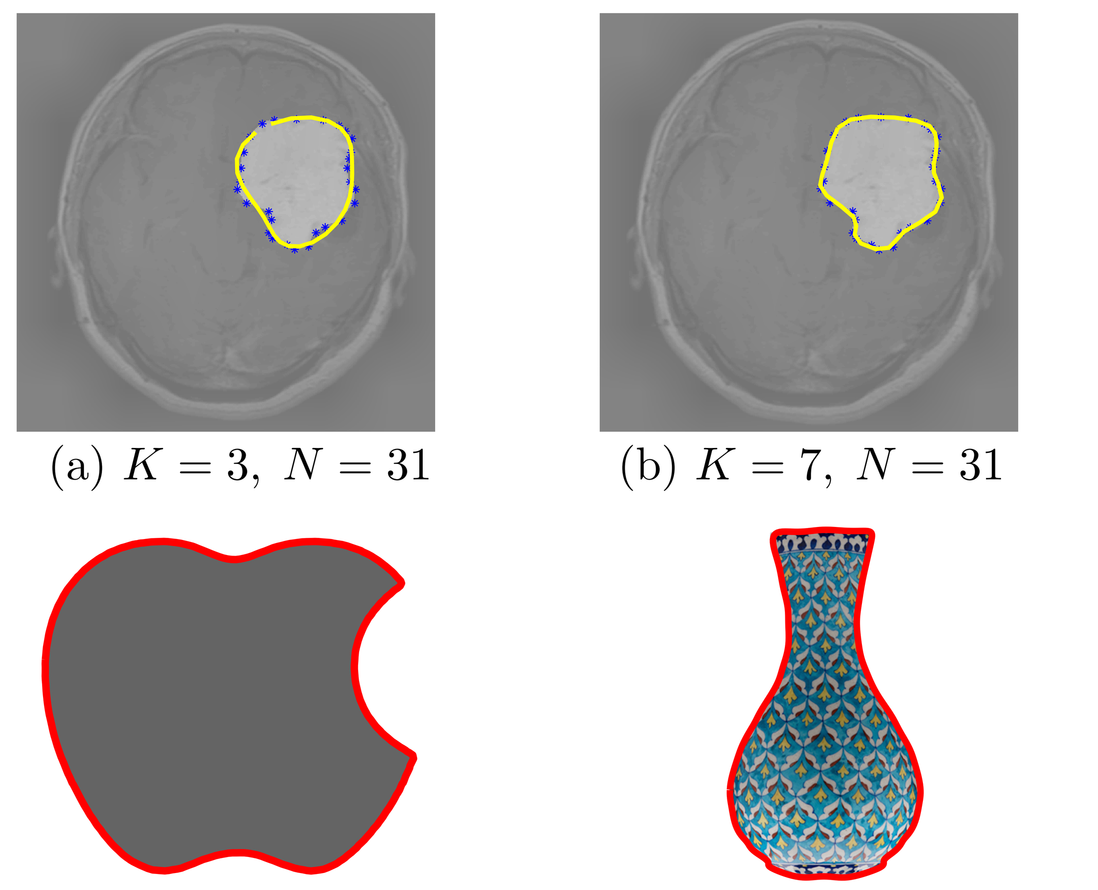


Figure 2: Reconstruction of shapes using the FRI-FD and direct methods. Model orders are $K = 2$ for (a) and (b), and $K = 3$ for (c)-(f). Reconstruction was performed using 60% of the measurements.

The FRI-FD method reliably reconstructs curves from partial measurements as long as the measurements are taken from the regions with high curvature. This aspect needs *further investigation.*

7. Application to Real Images



(a) $K = 3$, $N = 31$ (b) $K = 7$, $N = 31$

(c) $K = 100$, $N = 340$ (d) $K = 25$, $N = 345$

Figure 3: Outlining of (a)-(b) tumours in brain MR images and (c)-(d) different shapes reconstructed after Canny edge detection with model order K and from N samples.

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